Thermally Induced Nutational Body Motion of a Spinning Spacecraft with Flexible Appendages

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This paper deals with a thermally induced nutational body motion of a spinning spacecraft with flexible appendages. The explanation of this phenomenon is as follows: when this class of spacecraft is exposed to solar radiation, thermally induced bending vibrations of appendages occur at a spin frequency. These vibrations cause a periodic variation of the moments of inertia of the spacecraft at that frequency and this, in turn, produces self-excitation of nutational body motions through the instrumentality of the parametric excitation. The amplitude of the nutational body motion is determined by means of the method of averaging and the stability of the motion is examined in detail.

Introduction

CURRENT designs of spacecraft employ large, highly flexible appendages such as antennas or solar arrays. Increasing attention has been focused in recent years on the dynamics of a flexible spacecraft. The influence of vehicle nonrigidity on the spacecraft attitude stability has been investigated by many authors[1-4] using the energy sink method or the Liapunov direct method. On the other hand, several papers have been published to explain the anomalous behaviors of flexible spacecrafts. Vigneron and Boresi[5] have presented the analysis of the long term spin decay caused by the structural damping combined with the gravity field. In attempt to explain the observed spin decay phenomena of the spin stabilized satellites, Alouette I, II, and Explorer XX, Etkin and Hughes[6] have investigated the despin mechanism of flexible spacecraft due to the action of solar radiation. Furthermore, analytical studies of thermally induced vibration of flexible appendages have been presented by several authors.[7,8] However, these works are restricted only to appendage motions and not concerned with spacecraft dynamics.

The objective of this study is to predict that a class of spinning spacecraft with flexible appendages may exhibit an anomalous behavior, a steady nutational body motion which is caused by an interaction of flexible appendages with solar radiation. The problem to be analyzed is as follows: a spinning spacecraft which has flexible appendages is considered (Fig. 1). The appendages are assumed to lie in a plane normal to the spin axis. Solar radiation is assumed normal to the spin axis. When this class of spacecraft is exposed to solar radiation, vibrations of the appendages in a spin plane are induced at a spin frequency by solar heating. These vibrations cause a periodic variation of the moments of inertia of the spacecraft. Usually, the influence of this variation of the moments of inertia upon nutational body motions is small. However, if the spin velocity is approximately equal to twice the frequency of the nutational body motion (this means that the ratio of the spin moment of inertia to the transverse moment of inertia is nearly equal to $\frac{3}{2}$ for the vehicle in its undeformed shape) the amplitude of the nutational body motion builds up to larger values through the instrumentality of nonlinear parametric excitation. The method of averaging is applied to this problem to obtain the amplitude of the nutational body motion analytically, and the stability of the motion is examined in detail.

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Equations of Motion

Consider a symmetrical spinning spacecraft composed of a heavy central rigid body and lightweight flexible appendages (Fig. 1). The reference axes (X_1, X_2, X_3) are assumed to be parallel to the principal axes of the undeformed total configuration (X_3 axis coincides with the spin axis) and have the origin at the mass center of the undeformed total configuration. For an appendage i, an axis system (ζ_i, η_i, ξ_i) is defined so that the appendage is coincident with the ζ_i axis at the undeflected condition, and the ξ_i axis coincides with the spin axis. Let the angle of rotation from (X_1, X_2, X_3) to (ζ_i, η_i, ξ_i) be γ_i and let the angular velocity of the (X_1, X_2, X_3) axes have the components $(\omega_1, \omega_2, \omega_3)$ in the (X_1, X_2, X_3) reference frame.

In the present study, the mass center is assumed to remain fixed at the origin of the axis system (X_1, X_2, X_3). As mentioned above, thermally induced vibrations of the appendages in the spin plane cause a nutational body motion of the spacecraft. The nutational body motion, on the other hand, induces out of plane vibrations of the appendages. The out of plane vibrations can interact with the nutational body motion, but, in what follows, we shall neglect the out of plane vibrations for the sake of simplicity. This simplification does not alter the essential features of the phenomena discussed herein. Furthermore, we shall, for the moment, confine ourselves to the case where the effect of solar heating is so small that the induced vibrations of the appendages are small. Then, the total kinetic energy T takes the form,

$$2T = I(\omega_1{}^2 + \omega_2{}^2) + I_3\omega_3{}^2 + \sum_{i=1}^{N} \mu_i \int_o^{l_i} \left\{ \dot{u}_i{}^2 + 2\omega_3 \zeta_i \dot{u}_i + S2\gamma_i{}^2 \omega_1{}^2 \zeta_i u_i - S2\gamma_i \omega_2{}^2 \zeta_i u_i + \omega_3{}^2 \left[u_i{}^2 - \tfrac{1}{2}(l_i{}^2 - \zeta_i{}^2)\left(\frac{\partial u_i}{\partial \zeta_i}\right)^2 \right] - 2C2\gamma_i \omega_1 \omega_2 \zeta_i u_i \right\} d\zeta_i \qquad (1)$$

where I and I_3 are the moments of inertia of the undeformed total configuration about X_1 (or X_2), X_3 axes, respectively, u_i an in-plane deflection of an appendage i, l_i the length of the appendage i, μ_i the mass per unit length of the appendage i and $S2\gamma_i = \sin 2\gamma_i$, $C2\gamma_i = \cos 2\gamma_i$.

The elastic potential energy U which arises from the strain energy due to the appendage deformations is given by

$$2U = \sum_{i=1}^{N} B_i \int_o^{l_i} \left(\frac{\partial^2 u_i}{\partial \zeta_i{}^2}\right)^2 d\zeta_i \qquad (2)$$

where B_i is the bending stiffness of an appendage i. The energy dissipation which results from elastic deformations of the

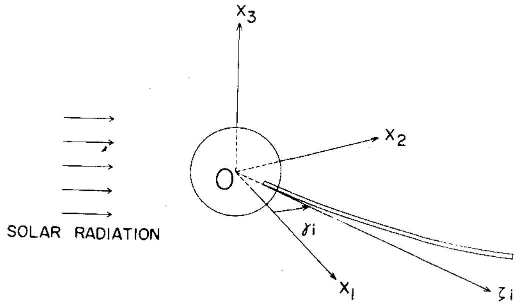

Fig. 1 Spacecraft configuration.

appendages is represented by Rayleigh's dissipation function F, which is given by

$$F = \sum_{i=1}^{N} \mu_i \delta_i \int_o^{l_i} \dot{u}_i^2 \, d\zeta_i \tag{3}$$

where δ_i is the damping ratio for an appendage i.

We shall represent the elastic deformations u_i by the following series:

$$u_i(\zeta_i, t) = l_i \sum_{n=1}^{\infty} P_{in}(t) E_n(\xi) \tag{4a}$$

where $E_n(\xi)$ are normal modes associated with a cantilever and P_{in} are corresponding generalized coordinates. The normal modes $E_n(\xi)$ are defined by

$$\begin{gathered}(d^4 E_n(\xi)/d\xi^4) - \lambda_n^4 E_n(\xi) = 0 \\ \xi = 0, \quad E_n(\xi) = (dE_n(\xi)/d\xi) = 0 \\ \xi = 1, \quad [d^2 E_n(\xi)/d\xi^2] = [d^3 E_n(\xi)/d\xi^3] = 0\end{gathered} \tag{5}$$

where λ_n^4 are the eigenvalues of the normal modes $E_n(\xi)$ and $\xi = \zeta_i/l_i$. In addition, they are normalized such that

$$\int_o^1 E_n(\xi) E_m(\xi) \, d\xi = \delta_{n,m} \tag{6}$$

where $\delta_{n,m}$ is Kronecker's delta. In this study, attention will be paid to the case in which the appendages are excited near the first resonance frequency. Then, the deflections, u_i, can be approximated by the first mode, i.e.,

$$u_i(\zeta_i, t) = l_i P_{i1}(t) E_1(\xi) \tag{4b}$$

Substituting Eq. (4b) into Eqs. (1–3) and neglecting the suffix 1, we obtain

$$\begin{aligned}2T = {} & I(\omega_1^2 + \omega_2^2) + I_3 \omega_3^2 + \sum_{i=1}^{N} \mu_i l_i^3 \times \\ & [\dot{P}_i^2 + 2\varepsilon\omega_3 \dot{P}_i + S2\gamma_i \omega_1^2 P_i - S2\gamma_i \varepsilon \omega_2^2 P_i + (1-\beta)\omega_3^2 P_i^2 - \\ & 2C2\gamma_i \varepsilon \omega_1 \omega_2 P_i]\end{aligned} \tag{7}$$

$$2U = \sum_{i=1}^{N} (\lambda^4 B_i / l_i) P_i^2 \tag{8}$$

$$F = \sum_{i=1}^{N} \mu_i l_i \delta_i \dot{P}_i^2 \tag{9}$$

where

$$\varepsilon = \int_o^1 \xi E(\xi) \, d\xi = -0.5688$$

$$\beta = \tfrac{1}{2} \int_o^1 (1 - \xi^2)[dE(\xi)/d\xi]^2 \, d\xi = 1.193$$

Since the coordinates $\omega_1, \omega_2, \omega_3$ are so-called quasi-coordinates, the corresponding equations of motion are

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_1}\right) + \omega_2 \left(\frac{\partial T}{\partial \omega_3}\right) - \omega_3 \left(\frac{\partial T}{\partial \omega_2}\right) &= N_1 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \omega_2}\right) + \omega_3 \left(\frac{\partial T}{\partial \omega_1}\right) - \omega_1 \left(\frac{\partial T}{\partial \omega_3}\right) &= N_2 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \omega_3}\right) + \omega_1 \left(\frac{\partial T}{\partial \omega_2}\right) - \omega_2 \left(\frac{\partial T}{\partial \omega_1}\right) &= N_3\end{aligned} \tag{10}$$

where N_1, N_2, N_3 are the external torque components about the X_1, X_2, X_3 axes, respectively. The Lagrange equations of motion for the generalized coordinates P_i take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{P}_i}\right) + \left(\frac{\partial F}{\partial \dot{P}_i}\right) - \left(\frac{\partial L}{\partial P_i}\right) = Q_i \tag{11}$$

where the Lagrangian L is given by $L = T - U$, Q_i are the generalized forces arising from external sources. The attitude motion of the spacecraft is affected by external torques and forces of various forms. The present paper considers nutational body motions induced by solar heating, so that other external torques and forces are neglected.

Solar heating produces a temperature difference across an appendage. The temperature distribution induces a thermal strain along the appendage length. Etkin and Hughes[5] state that, on the assumption that solar radiation is normal to the spin axis, the steady periodic thermal bending moment with the spin frequency at any section of an appendage i approximated by

$$M_{i_1} = 0, \quad M_{i_2} = 0, \quad M_{i_3} = f_{oi} \cos(\tau + \tau_i) \tag{12}$$

where M_{i1}, M_{i2}, M_{i3} are the components of the thermal bending moment about ζ_i, η_i, ξ_i axes, respectively, f_{oi} is a constant, $\tau_i = \tau_o + \gamma_i$ (τ_o is a constant phase lag), and $d\tau/dt = \omega_3$. The work done by the thermal bending moment in an arbitrary displacement δu_i takes the form

$$\delta W = \sum_{i=1}^{N} f_{oi} \cos(\tau + \tau_i) \int_o^{l_i} \left(\frac{\partial^2(\delta u_i)}{\partial \zeta_i^2}\right) d\zeta_i \tag{13}$$

Upon substitution of Eq. (4a) into Eq. (13) and neglecting the suffix 1, we find

$$\delta W = \sum_{i=1}^{N} f_{oi} \cos(\tau + \tau_i) \frac{E'(1)}{l_i} \delta P_i \tag{14}$$

Then, the generalized forces Q_i arising from the effect of solar heating are obtained from Eqs. (14) as

$$Q_i = f_{oi} \cos(\tau + \tau_i)(E'(1)/l_i) \tag{15}$$

On the other hand, the action of the solar radiation pressure produces torques on the spacecraft. The torques, however, are so small that we neglect the torques in the following analysis. Then, it follows:

$$N_1 = 0, \quad N_2 = 0, \quad N_3 = 0 \tag{16}$$

Substituting Eqs. (7–9, 15, and 16) into Eqs. (10) and (11), we obtain a system of equations of motion as follows:

$$\begin{aligned}\dot{\omega}_1 + \alpha\omega_2 = {} & -\sum_{i=1}^{N} \frac{\varepsilon \mu_i l_i^3}{I} [\omega_2 \dot{P}_i + S2\gamma_i (P_i \dot{\omega}_1 + \dot{P}_i \omega_1) - \\ & C2\gamma_i (\dot{P}_i \omega_2 + P_i \dot{\omega}_2) + C2\gamma_i \omega_1 \omega_3 P_i + S2\gamma_i \omega_2 \omega_3 P_i \\ \dot{\omega}_2 - \alpha\omega_1 = {} & \sum_{i=1}^{N} \frac{\varepsilon \mu_i l_i^3}{I} [\omega_1 \dot{P}_i + C2\gamma_i (\dot{P}_i \omega_1 + P_i \dot{\omega}_1) + \\ & S2\gamma_i (\dot{P}_i \omega_2 + P_i \dot{\omega}_2) - S2\gamma_i \omega_1 \omega_3 P_i + C2\gamma_i \omega_2 \omega_3 P_i] \\ \dot{\omega}_3 = {} & -\sum_{i=1}^{N} (\varepsilon \mu_i l_i^3 / I_3) \dot{P}_i \\ \ddot{P}_i + 2\delta_i \dot{P}_i + k_i^2 P_i = {} & -\varepsilon \dot{\omega}_3 + \frac{\varepsilon}{2} (S2\gamma_i \omega_1^2 - 2C2\gamma_i \omega_1 \omega_2 - \\ & S2\gamma_i \omega_2^2) + (f_{oi} E'(1)/\mu_i l_i^4) \cos(\tau + \tau_i) \\ \dot{\tau} = {} & \omega_3\end{aligned} \tag{17}$$

where

$$\alpha = \left(\frac{I_3}{I} - 1\right)\omega_3, \qquad k_i^2 = \frac{\lambda^4 B_i}{\mu_i l_i^4} + (\beta - 1)\omega_3^2$$

We introduce a new complex variable a as follows:

$$\omega_1 = (a e^{i\tau/2} + a^* e^{-i\tau/2}), \quad \omega_2 = -i(a e^{i\tau/2} - a^* e^{-i\tau/2}) \tag{18}$$

where a^* is a complex conjugate of a. Substituting Eqs. (18) into Eqs. (17), we have

$$\dot{a} = i\left(\alpha - \frac{\omega_3}{2}\right)a + \varepsilon \sum_{i=1}^{N} \frac{\mu_i l_i^3}{2I}(2ia\dot{P}_i + 2ia^*\dot{P}_i e^{i2\gamma_i} e^{-i\tau} - \omega_3 a^* P_i e^{i2\gamma_i} e^{-i\tau} + 2i\dot{a}^* P_i e^{i2\gamma_i} e^{-i\tau}) \tag{19a}$$

$$\dot{\omega}_3 = -\varepsilon \sum_{i=1}^{N} \frac{\mu_i l_i^3}{I_3}\ddot{P}_i \tag{19b}$$

$$\ddot{P}_i + 2\delta_i \dot{P}_i + k_i^2 P_i = -\varepsilon\dot{\omega}_3 + i\varepsilon(a^2 e^{-i2\gamma_i} e^{i\tau} - a^{*2} e^{i2\gamma_i} e^{-i\tau}) + \frac{f_{oi}E'(1)}{2\mu_i l_i^4}(e^{i(\tau+\tau_i)} + e^{-i(\tau+\tau_i)}) \tag{19c}$$

$$\dot{\tau} = \omega_3 \tag{19d}$$

Since we suppose that the effect of solar heating is so small that the induced vibrations of the appendages are small, we can write P_i and f_{oi} in the form

$$P_i = \varepsilon\hat{P}_i, \qquad f_{oi} = \varepsilon\hat{f}_{oi} \tag{20}$$

Substituting these into Eqs. (19), we obtain

$$\dot{a} = i\left(\alpha - \frac{\omega_3}{2}\right)a + \varepsilon^2 \sum_{i=1}^{N} \frac{\mu_i l_i^3}{2I} \times (2ia\dot{\hat{P}}_i + 2ia^*\dot{\hat{P}}_i e^{i2\gamma_i} e^{-i\tau} - \omega_3 a^*\hat{P}_i e^{i2\gamma_i} e^{-i\tau} + 2i\dot{a}^*\hat{P}_i e^{i2\gamma_i} e^{-i\tau}) \tag{21a}$$

$$\dot{\omega}_3 = -\varepsilon^2 \sum_{i=1}^{N} \frac{\mu_i l_i^3}{I_3}\ddot{\hat{P}}_i \tag{21b}$$

$$\ddot{\hat{P}}_i + 2\delta_i \dot{\hat{P}}_i + k_i^2 \hat{P}_i = -\dot{\omega}_3 + i(a^2 e^{-i2\gamma_i} e^{i\tau} - a^{*2} e^{i2\gamma_i} e^{-i\tau}) + \frac{\hat{f}_{oi}E'(1)}{2\mu_i l_i^4}(e^{i(\tau+\tau_i)} + e^{-i(\tau+\tau_i)}) \tag{21c}$$

$$\dot{\tau} = \omega_3 \tag{21d}$$

Let us suppose the spin velocity of the spacecraft is approximately equal to twice the angular velocity of the nutational body motion, i.e.,

$$2\alpha \cong \omega_3 \tag{22a}$$

This condition means that

$$I_3/I \cong \tfrac{3}{2} \tag{22b}$$

Then, we can conclude from Eqs. (21) that the variables (a, ω_3) are slowly varying functions because $(\dot{a}, \dot{\omega}_3) \sim O(\varepsilon)$, while the variables (P_i, τ) vary relatively rapidly. Hence, approximate solutions of Eqs. (21) can be obtained by the method of averaging applied to a system containing both slow and rapid motions.

Briefly, the method of averaging applied to a system with rapid and slow motions is as follows:[9] Let us write such system in the following form:

$$\dot{x} = \varepsilon X(x, y, t, \varepsilon) \tag{23a}$$

$$\dot{y} = Y(x, y, t, \varepsilon) \tag{23b}$$

where $x = (x_1, \ldots, x_n)$, $X = (X_1 \ldots, X_n)$ and $y = (y_1 \ldots, y_m)$, $Y = (Y_1 \ldots, Y_m)$ are n- and m-dimensional vector functions, respectively, and ε is a small quantity. The variables x_i are slowly varying since $\dot{x}_i \sim \varepsilon$, while the variables y_i are rapidly varying since $\dot{y}_i \sim 1$. Together with the system (23) we shall also consider the degenerate system:

$$x = \text{const} \tag{24a}$$

$$\dot{y} = Y_o(x, y, t) = Y(x, y, t, 0) \tag{24b}$$

When a solution is given by the form

$$x = \bar{x} + \sum_{n=1}^{\infty} \varepsilon^n U_n(x, y, t), \quad y = \bar{y} + \sum_{n=1}^{\infty} \varepsilon^n V_n(x, y, t) \tag{25}$$

where U_i, V_i are n- and m- dimensional vector functions, respectively, the equation of the first approximation to x is obtained in the form

$$\dot{\bar{x}} = \varepsilon \bar{X}_o(\bar{x}) \tag{26}$$

$$\bar{X}_o = \lim_{T\to\infty} \frac{1}{T}\int_o^T X(\bar{x}, \zeta(\bar{x}, t), t, 0)\, dt \tag{27}$$

where $\zeta(\bar{x}, t)$ is an integral curve of the system (24b). The solution of Eq. (26) is expected to approximate, with an error ε, to the solution of the original system (23a) on an interval $t \sim \varepsilon^{-1}$.

Approximate Solutions and Their Stability

The method as outlined can be applied to Eqs. (21) to obtain the first approximate solutions. Equations corresponding to Eqs. (24) are given by

$$a = \bar{a} \tag{28a}$$

$$\omega_3 = \bar{\omega}_3 \tag{28b}$$

$$\ddot{\hat{P}}_i + 2\delta_i \dot{\hat{P}}_i + \bar{k}_i^2 \hat{P}_i = i(\bar{a}^2 e^{-i2\gamma_i} e^{i\tau} - \bar{a}^{*2} e^{i2\gamma_i} e^{-i\tau}) + (\hat{f}_{oi} E'(1)/2\mu_i l_i^4)(e^{i(\tau+\tau_i)} + e^{-i(\tau+\tau_i)}) \tag{28c}$$

$$\dot{\tau} = \bar{\omega}_3 \tag{28d}$$

where

$$\bar{k}_i^2 = (\lambda^4 B_i/\mu_i l_i^4) + (\beta - 1)\bar{\omega}_3^2$$

From Eq. (28d)

$$\tau = \bar{\omega}_3 t \tag{29}$$

The steady-state solutions of Eqs. (28c) are given by

$$\hat{P}_i = \frac{i\bar{a}^2 e^{-i2\gamma_i} e^{i\bar{\omega}_3 t}}{\bar{k}_i^2 + 2i\delta_i\bar{\omega}_3 - \bar{\omega}_3^2} + \frac{\hat{f}_{oi}E'(1)}{2\mu_i l_i^4}\frac{e^{i(\bar{\omega}_3 t + \tau_i)}}{\bar{k}_i^2 + 2i\delta_i\bar{\omega}_3 - \bar{\omega}_3^2} + \text{complex conjugate part.} \tag{30}$$

Substituting Eqs. (29) and (30) into Eqs. (21a) and (21b) and averaging as in Eq. (27), we obtain an averaged system of equations as follows:

$$\dot{\bar{a}} = i\left(\bar{\alpha} - \frac{\bar{\omega}_3}{2}\right)\bar{a} - \varepsilon^2 \sum_{i=1}^{N}\frac{\mu_i l_i^3}{2I} \times \left(\frac{3i\bar{\omega}_3\bar{a}^2\bar{a}^*}{\bar{k}_i + 2i\delta_i\bar{\omega}_3 - \bar{\omega}_3^2} + \frac{\hat{f}_{oi}E'(1)}{2\mu_i l_i^4}\frac{3\bar{\omega}_3\bar{a}^* e^{i3\tau_i}}{\bar{k}_i^2 + 2i\delta_i\bar{\omega}_3 - \bar{\omega}_3^2}\right) \tag{31a}$$

$$\dot{\bar{\omega}}_3 = 0 \tag{31b}$$

where

$$\bar{\alpha} = (I_3/I - 1)\bar{\omega}_3$$

Eq. (31b) yields

$$\bar{\omega}_3 = \omega_o \tag{32}$$

By the use of this equation, Eq. (29) becomes

$$\tau = \omega_o t \tag{33}$$

Putting

$$a = (x + iy)\omega_o \tag{34}$$

and introducing a new independent variable

$$\tau = \omega_o t$$

we obtain from Eq. (31a)

$$x' = X(x, y), \qquad y' = Y(x, y) \tag{35}$$

where

$$X(x, y) = h_o y + h_{1r} x + h_{1i} y + r^2(h_{2r} x - h_{2i} y)$$

$$Y(x, y) = -h_o x + h_{1i} x - h_{1r} y + r^2(h_{2r} y + h_{2i} x)$$

$$h_o = \left(\frac{1}{2} - \frac{\alpha_o}{\omega_o}\right)$$

$$h_{1r} = -\varepsilon^2 \sum_{i=1}^{N} \frac{\hat{f}_{oi}E'(1)}{4l_i I}\frac{(k_{io}^2 - \omega_o^2)C3\tau_i + 2\delta_i\omega_o S3\tau_i}{(k_{io}^2 - \omega_o^2)^2 + (2\delta_i\omega_o)^2}$$

$$h_{1i} = -\varepsilon^2 \sum_{i=1}^{N} \frac{\hat{f}_{oi}E'(1)}{4l_i I}\frac{(k_{io}^2 - \omega_o^2)S3\tau_i - 2\delta_i\omega_o C3\tau_i}{(k_{io}^2 - \omega_o^2)^2 + (2\delta_i\omega_o)^2}$$

$$h_{2r} = -\varepsilon^2 \sum_{i=1}^{N} \frac{3\mu_i l_i^3\omega_o^2}{2I}\frac{2\delta_i\omega_o}{(k_{io}^2 - \omega_o^2)^2 + (2\delta_i\omega_o)^2}$$

$$h_{2i} = -\varepsilon^2 \sum_{i=1}^{N} \frac{3\mu_i l_i^3\omega_o^2}{2I}\frac{(k_{io}^2 - \omega_o^2)}{(k_{io}^2 - \omega_o^2)^2 + (2\delta_i\omega_o)^2}$$

$$\alpha_o = \left(\frac{I_3}{I} - 1\right)\omega_o, \quad k_{io}^2 = \frac{\lambda^4 B_i}{\mu_i l_i^4} + (\beta - 1)\omega_o^2, \quad r^2 = x^2 + y^2$$

The primes denote the differentiation with respect to τ. It may be noted, from Eqs. (18) and (34), the quantity r is proportional to the amplitude of a nutational body motion, $(\omega_1{}^2+\omega_2{}^2)^{1/2}/\omega_o$. By the use of the method of averaging the system of Eqs. (21) has been reduced to the system of Eqs. (35). This system of equations is that of two ordinary differential equations of the first order, so that we can successfully apply the topological method to obtain the overall views of the motion.

Let us now consider in more detail the steady state where the quantities x and y in Eqs. (35) are constant:

$$X(x,y)=0, \qquad Y(x,y)=0 \tag{36}$$

Substitution of these conditions into Eqs. (35) leads to the determination of the steady-state amplitude of $r_o[=(x_o{}^2+y_o{}^2)^{1/2}]$ as follows:

$$r_o{}^2=0 \tag{37a}$$

$$r_o{}^2=\frac{h_o h_{2i}}{(h_{2r}{}^2+h_{2i}{}^2)}\pm \left[(h_o h_{2i})^2+(h_{1r}{}^2+h_{1i}{}^2-h_o{}^2)\times(h_{2r}{}^2+h_{2i}{}^2)\right]^{1/2} \tag{37b}$$

and the components x_o, y_o of the amplitude r_o are

$$x_o{}^2=\frac{r_o{}^2}{1+(Q/p)^2}, \qquad y_o{}^2=\frac{r_o{}^2}{1+(P/Q)^2} \tag{38}$$

where

$$P=h_{1r}+r_o{}^2h_{2r}$$
$$Q=h_o+h_{1i}-r_o{}^2h_{2i}$$

As an example, a symmetrical spacecraft having three equal appendages is investigated. The appendages make an angle of 120° with each other. Figure 2 shows the relationship between h_o and $r_o{}^2$ in the case where $h_{1r}=0.110$, $h_{1i}=0.441$, $h_{2r}=1.51$, $h_{2i}=-3.77$.

Then, let us investigate the stability of the steady-state solutions, Eqs. (37). To establish necessary and sufficient conditions for the stability of these solutions, we must construct the variational equations about these solutions. The variational coordinates are characterized by the symbols δx and δy. From Eqs. (35) we obtain the variational equations as follows:

$$\delta x'=a_1\,\delta x+a_2\,\delta y, \qquad \delta x'=b_1\,\delta x+b_2\,\delta y \tag{39}$$

where

$$a_1=\left(\frac{\partial X}{\partial x}\right)_{\substack{x=x_o\\y=y_o}}=h_{1r}+r_o{}^2h_{2r}+2h_{2r}x_o{}^2-2x_oy_oh_{2i}$$

$$a_2=\left(\frac{\partial X}{\partial y}\right)_{\substack{x=x_o\\y=y_o}}=(h_o+h_{1i})-r_o{}^2h_{2i}-2h_{2i}y_o{}^2+2x_oy_oh_{2r}$$

$$b_1=\left(\frac{\partial Y}{\partial x}\right)_{\substack{x=x_o\\y=y_o}}=(-h_o+h_{1i})+r_o{}^2h_{2i}+2h_{2i}x_o{}^2+2x_oy_oh_{2r}$$

$$b_2=\left(\frac{\partial Y}{\partial y}\right)_{\substack{x=x_o\\y=y_o}}=-h_{1r}+r_o{}^2h_{2r}+2h_{2r}y_o{}^2+2x_oy_oh_{2i}$$

The characteristic equation of the system defined by Eq. (39) is

$$\lambda^2-(a_1+b_2)\lambda+(a_1b_2-a_2b_1)=0 \tag{40}$$

First, we shall consider the stability of the solution (37a). The characteristic equation corresponding to this solution becomes

$$\lambda^2-(h_{1r}{}^2+h_{1i}{}^2-h_o{}^2)=0 \tag{41}$$

As the stability criterion, the roots of the characteristic equation must not have a positive real part, we have

$$h_{1r}{}^2+h_{1i}{}^2-h_o{}^2\leqq 0 \tag{42}$$

If the condition (42) is satisfied, the characteristic roots are purely imaginary. Since the condition that the characteristic roots are purely imaginary is not sufficient for the stability of the solution, the stability condition must be examined in more detail

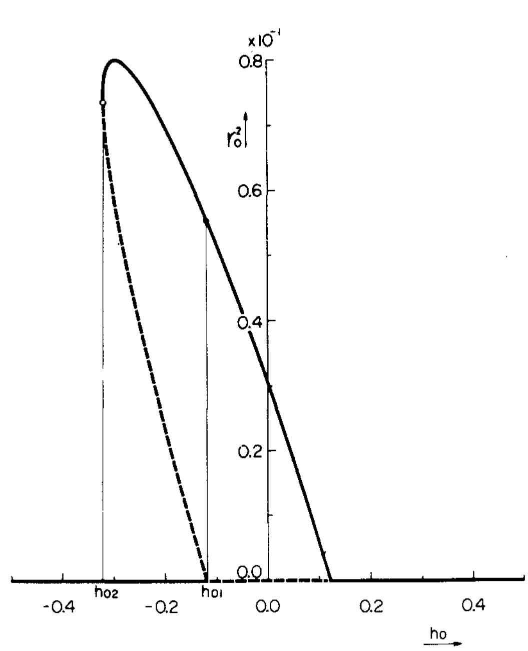

Fig. 2 Frequency response curves of nutational body motion.

from the consideration of the type of the singularity at the origin of the system of Eqs. (35), which is correlated with the solution (37a). By virtue of the procedure due to Poincare[10] it is clarified that if $h_{2r}<0$, the singularity is a stable focus, if $h_{2r}>0$, the singularity is an unstable focus, if $h_{2r}=0$, the singularity is a center. The stability conditions are, therefore, set up as follows:

$$h_{2r}\leqq 0 \tag{43a}$$

$$h_{1r}{}^2+h_{1i}{}^2-h_o{}^2\leqq 0 \tag{43b}$$

Next, we shall consider the stability of the solution (37b). The characteristic equation corresponding to this solution is given by

$$\lambda^2-4h_{2r}r_o{}^2\lambda+4r_o{}^2[h_{2r}{}^2r_o{}^2-(h_o-r_o{}^2h_{2i})h_{2i}]=0 \tag{44}$$

The stability condition is derived by the Routh-Hurwitz criterion;

$$h_{2r}<0 \tag{45a}$$

$$(h_{2r}{}^2+h_{2i}{}^2)r_o{}^2-h_oh_{2i}>0 \tag{45b}$$

It may be noted that the conditions (43a), (45a) are always fulfilled, since we are concerned with a damped mechanical system. Let these conditions for stability be represented in Fig.2. In Fig. 2, the dashed parts of the response curves are unstable and the solid lines of the response curves are stable. These peculiar conditions for stability of nonlinear external resonance cause the appearance of the so-called jumping phenomenon.[11] In Fig. 2, with increasing h_o, it is observed that the nutational body motion starts abruptly with a finite amplitude for $h_o=h_{o1}$ and decreases smoothly for $h_o>h_{o1}$. On the contrary, for decreasing h_o it is observed that the phenomenon is different; namely, for $h_o=h_{o1}$ the nutational body motion does not disappear and it jumps down and disappears at $h_o=h_{o2}$.

Domains of Attraction

We shall here investigate the relationship between the initial conditions and the resulting steady-state solutions of a system governed by Eqs. (35). For this purpose it is useful to investigate the integral curves of the following equation derived from Eqs. (35), i.e.,

$$(dy/dx)=[Y(x,y)/X(x,y)] \tag{46}$$

A singular point, for which $X(x,y)=0$ and $Y(x,y)=0$, corresponds to a steady-state solution of Eqs. (35). Let us trace the integral curves for certain typical cases on the x,y plane. The special case considered here are characterized by the following values of the system parameters. Case 1: $h_o=-0.25$,

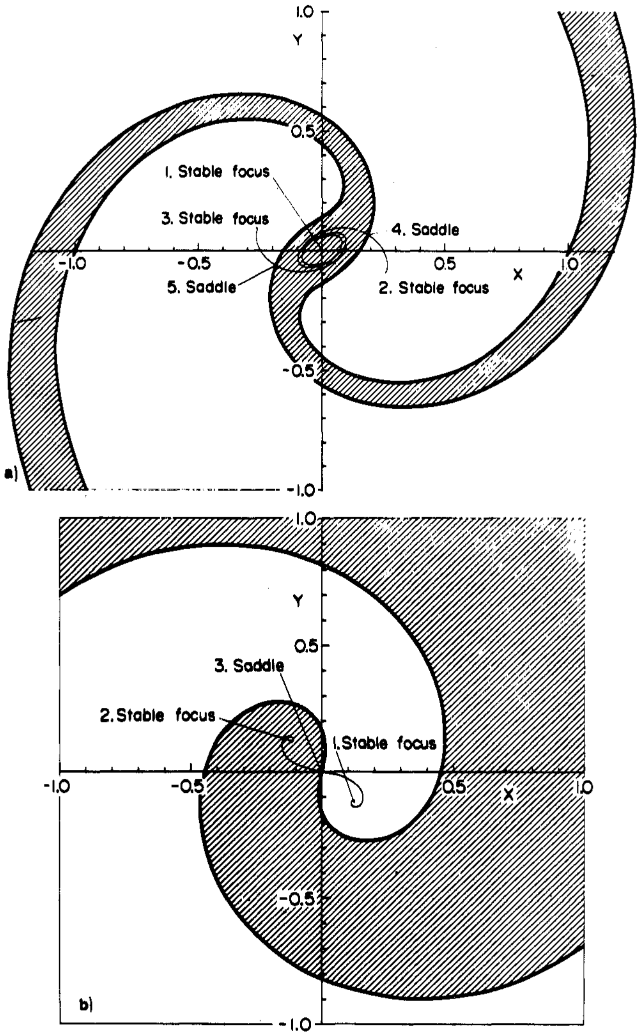

Fig. 3 Integral curves of Eq. (46) in the xy plane: a) case 1; b) case 2.

$h_{1r} = 0.110$, $h_{1i} = -0.0441$, $h_{2i} = -1.51$, $h_{2r} = -3.77$. Case 2: $h_o = 0.0$, $h_{1r} = 0.110$, $h_{1i} = -0.0441$, $h_{2i} = -1.51$, $h_{2r} = -3.77$.

With the aid of the numerical integration of Eq. (46), the integral curves for these cases are drawn in Fig. 3a and 3b, respectively. The singularities in Figs. 3a and 3b are listed in Table 1. The integral curve which tends to a saddle point with increasing the time t is a separatrix which divides the coordinate plane into two regions, where any initial condition results in passage to a particular class of steady-state solutions. From these figures the relationship between initial conditions and the resulting steady-state solutions is easy to understand: in Fig. 3b, a nutational body motion started with any initial conditions in the shaded region tends ultimately to the singularity of point 2, whereas a nutational body motion started from the unshaded region tends to the singularity of point 1. Finally, let us investigate conservative systems. Although these systems are somewhat unrealistic, the characteristics of the integral curves are interesting. Putting $\delta_i = 0$ in Eqs. (35), we obtain

$$(dy/dx) = [Y_o(x, y)/X_o(x, y)] \quad (47)$$

where

$$\begin{aligned} X_o(x, y) &= h_o y + h_{1r} x + h_{1i} y - r^2 y h_{2i} \\ Y_o(x, y) &= -h_o x + h_{1i} x - h_{1r} y + r^2 x h_{2i} \end{aligned} \quad (48)$$

from which we obtain

$$X_o(x, y)\, dy - Y_o(x, y)\, dx = 0 \quad (49)$$

Since from Eqs. (48)

$$\partial X_o/\partial x + \partial Y_o/\partial y = 0 \quad (50)$$

Equation (47) becomes an exact differential equation, and the complete integral is given by

$$c = \frac{h_o}{2}(x^2+y^2) - \frac{h_{1i}}{2}(x^2-y^2) - h_{2i}\left(\frac{x}{2}+\frac{y}{2}\right)^2 + h_{1r} xy \quad (51)$$

where c is a constant of integration. The integral curves of Eq. (49) are readily obtained by plotting Eq. (51). Figure 4 shows the integral curves of Eq. (49), where the system parameter values are given by $h_o = -0.25$, $h_{1r} = 0.128$, $h_{1i} = 0.0$, $h_{2r} = 0.0$, $h_{2i} = -4.37$. It is noted that, in a conservative system, each integral curve forms a closed trajectory and does not tend to a stable singularity. This means that the amplitude of the nutational body motion exhibits slowly periodic changes.

Table 1 Singular points in Figs. 3a and 3b

Singular point	x_o	y_o	λ	Classification
Fig. 3a				
1	0	0	± 0.221	Stable focus
2	0.259	-0.0909	$-0.228 \pm i0.198$	Stable focus
3	-0.259	0.0909	$-0.228 \pm i0.198$	Stable focus
4	0.186	0.0653	-0.117 ± 0.246	Saddle
5	-0.186	-0.0653	-0.117 ± 0.246	Saddle
Fig. 3b				
1	0.121	-0.121	$-0.883 \pm i0.220$	Stable focus
2	-0.121	0.121	$-0.883 \pm i0.220$	Stable focus
3	0	0	± 0.119	Saddle

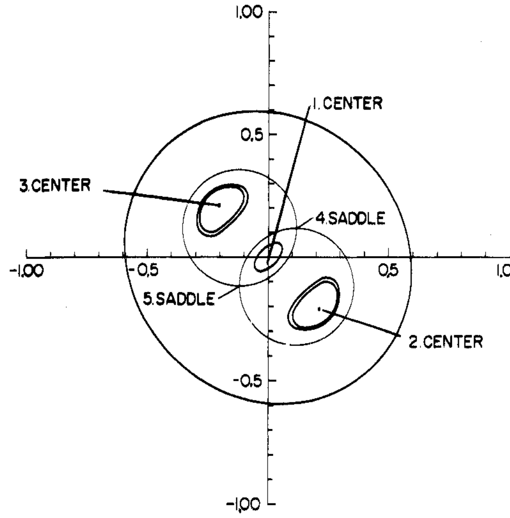

Fig. 4 Integral curves of Eq. (47) in the xy plane.

Conclusions

It has been demonstrated that a spacecraft with flexible appendages may exhibit thermally induced nutational body motions, as illustrated in Fig. 2. The amplitude of the nutational body motion is determined analytically and the stability of the motion is examined in detail.

This phenomenon is considered an external resonance phenomenon of a nonlinear mechanical system with several degrees of freedom. In the preceeding work, the use of the method of averaging is seen to reduce the fundamental equations to a system of two ordinary nonlinear differential equations of the first order, so that the general nature of the mechanical system is easily obtained.

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Analysis of Multiloop, Multirate Sampled-Data Systems

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Based upon new identities between z-transforms at a basis rate, z-transforms at faster rates, and modified z-transforms, the equivalence between the frequency decomposition method and the switch decomposition method is precisely presented so that results of one method are easily obtained from results of the other. Next, a method is developed for determining the closed loop transfer function of multiloop, multirate sampled-data systems with noninteger ratio sampling rates. Previously, this process involved solving a complex system of equations with rational polynomial coefficients. Herein, this is avoided by introducing a systematic decomposition of matrix operators which naturally arise from the switch decomposition method. The matrix operators are simplified by introducing the shifted transforms of signals sampled at one of the faster rates.

Introduction

THE use of digital computers in the guidance and control of missiles and space vehicles motivates the use of sampled data analysis methods. Such computers generally close several loops in the guidance and control tasks. Some of the guidance and control tasks can be performed more slowly than others while the over-all system meets all performance requirements. The use of multiple sampling rates can result in reduced computer capacity.

Previously developed methods of analysis of multiloop, multirate sampled-data systems are quite complex for systems which appear simple at first glance. Coffey and Williams[1] introduced a frequency-domain decomposition method for stability analysis of multiloop, multirate systems. In the general case, this frequency decomposition method numerically evaluated complex characteristic determinants for performing Nyquist stability analysis. Their analysis approach was a considerable improvement over Kranc's[2] switch decomposition method which was "state-of-the-art" up to that time. The switch decomposition method was based upon the frequency domain work of Sklansky[3] and was implemented by replacing samplers operating at various rates by an equivalent system of switches operating at a single rate. Jury[4] derived input-output relations for simple, open loop systems via both the frequency decomposition and switch decomposition methods and stated that the two forms are equivalent. The equivalence of the two forms could not be shown at that time due to a lack of invertible equivalence relations between the modified z-transform and the shifted argument, standard z-transforms. Boykin and Frazier[5] derived such equivalence relations and used these to develop a method of spectral factorization of the matrix operators which naturally arise in the switch decomposition method. In the case of sampling rates with integer ratios this spectral factorization enables the determination of closed loop transfer functions without ad hoc solutions of a system of equations with rational polynomial coefficients. The output was determined by inverting a spectrally factored matrix and was a one-step process.

This paper shows the equivalence of the frequency decomposition and switch decomposition methods through the equivalence relations and develops a method of determining closed loop transfer functions or characteristic equations of multiloop, multirate noninteger ratio sampled-data systems without evaluating

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